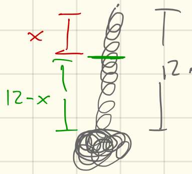
Quiz 3

Solutions

1. A chain lying on the ground is 15 m long and its mass is 100 kg. How much work is required to raise one end of the chain to a height of 12m?



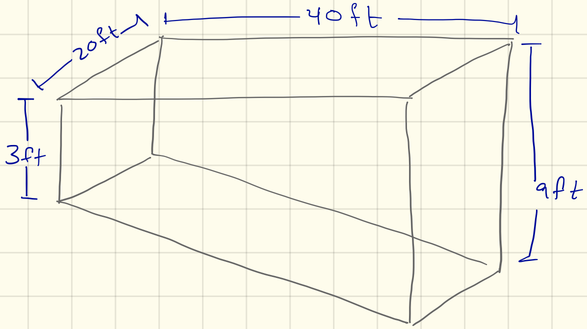
Weight per meter: $\frac{100\text{ kg}}{15\text{m}} \cdot 9.8\,\text{m}/\text{s}^2 = \frac{20 \cdot 9.8}{3}\ \text{N}/\text{m}$

$$\text{Work} = \int_0^{12} \frac{20 \cdot 9.8}{3}(12-x)\,dx$$

$$= \frac{20 \cdot 9.8}{3}\left(12x - \frac{x^2}{2}\right)\Big|_0^{12}$$

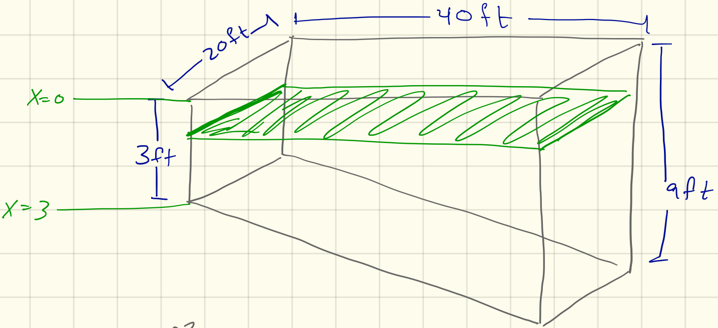
$$= \frac{20 \cdot 9.8}{3}\left(144 - \frac{144}{2}\right)\ \text{J}$$

2. A swimming pool is 20 ft wide and 40 ft long. Its bottom is an inclined plane the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, estimate the hydrostatic force on the shallow end, the deep end, one of the sides, and the bottom of the pool.



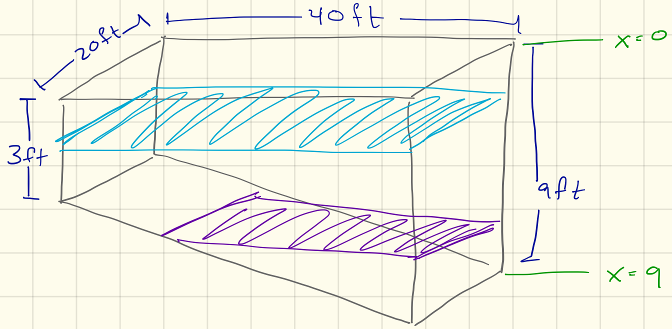
Weight of water: 62.5 lb/ft^3

(a) Shallow end



$$F = \int_0^3 62.5 \cdot 20 \, x \, dx = 5625 \text{ lbs.}$$

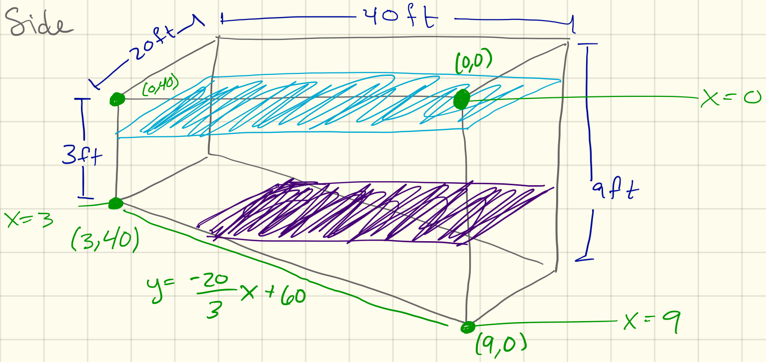
(b) Deep end



Note that the length of the sheet does not affect hydrostatic force.

$$F = \int_0^9 62.5 \cdot 20 \cdot x \, dx = 50625 \text{ lbs}$$

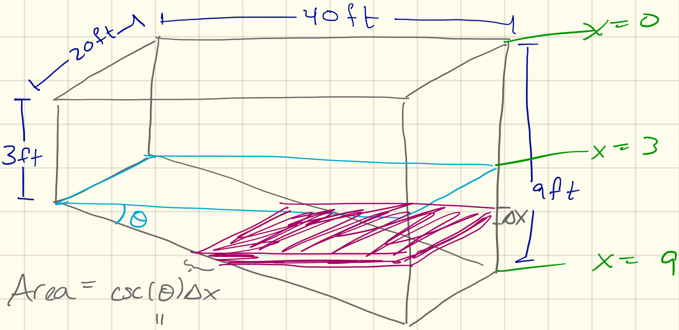
(c) Side



$$F = \overbrace{\int_0^3 62.5 \cdot 40x \, dx}^{\text{Rectangle part of side}} + \overbrace{\int_3^9 62.5 \cdot \left(-\frac{20}{3}x + 60\right)x \, dx}^{\text{triangle part of side}}$$

$$= 48750 \text{ lbs}$$

(d) Bottom



Area $= \csc(\theta)\Delta x$

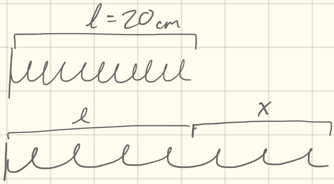
$$= \frac{\sqrt{409}}{3}\Delta x$$

pressure $= 62.5x$

$$F = \int_3^9 62.5 \cdot x \cdot 20 \cdot \frac{\sqrt{409}}{3}\,dx$$

$= 303356$ lb.

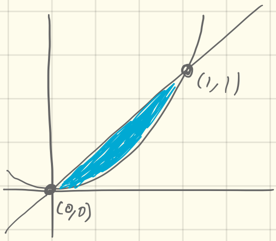
3. A spring has a natural length of 20cm. If a 25N force is required to keep it stretched to a length of 30cm, how much work is required to stretch it from 20cm to 25cm?



$$25 = k(0.1)$$
$$k = 250$$

$$F = \int_0^{0.05} 250x\,dx = \frac{250}{2} x^2 \Big|_0^{0.05} = \frac{250}{2} \cdot 0.05^2 \text{ N}$$

4. Find the centroid of the region bounded by the line $y=x$ and the parabola $y=x^2$.



$$A = \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\bar{x} = \frac{1}{A}\int_0^1 x\left(x - x^2\right)dx$$

$$= 6\int_0^1 x^2 - x^3\,dx$$

$$= 6\left(\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1\right)$$

$$= 6\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 6\left(\frac{1}{12}\right)$$

$$= \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left((x)^2 - (x^2)^2\right)dx$$

$$= 3\int_0^1 x^2 - x^4\,dx$$

$$= 3\left(\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1\right)$$

$$= 3\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 3\left(\frac{1}{12}\right)$$

$$= \frac{1}{4}$$

Centroid: $\left(\frac{1}{2}, \frac{1}{4}\right)$

5. For each condition below, give an example of a sequence $\{a_n\}$ with the desired property and justify that your sequence satisfies the given property.

(a) Converges to 0

$$\left\{\frac{1}{n}\right\} \qquad \lim_{n\to\infty} \frac{1}{n} = 0$$

(b) Converges but not to 0

$$\{1\} \qquad \lim_{n\to\infty} 1 = 1$$

(c) Diverges and bounded

$$\{(-1)^n\} \quad \lim_{n\to\infty} (-1)^n \text{ does not exist}$$

$$-1 \leq a_n = (-1)^n \leq 1 \quad \text{for all } n$$

(d) Diverges, unbounded, $\lim_{n\to\infty} a_n \neq \infty$ or $-\infty$

$$\{(-1)^n n\} \qquad \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} 2n = \infty$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} -(2n+1) = -\infty$$

So $\{a_n\}$ diverges and $\lim_{n\to\infty} a_n$ is not ∞ or $-\infty$.

6. Classify each statement as "always true," "sometimes true," or "never true". Justify your claim.

(a) The sequence $\{a_n\}$ converges to zero and $\sum a_n$ converges

Sometimes true: $a_n = \frac{1}{n}$ $\lim_{n\to\infty} a_n = 0$ but $\sum \frac{1}{n}$ diverges

$b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} b_n = 0$ and $\sum \frac{1}{n^2}$ converges

(b) The sequence $\{a_n\}$ converges to 1 and the series $\sum a_n$ converges.

Never true: If $\lim_{n\to\infty} a_n = 1$, the series $\sum a_n$ diverges by the divergence test.

(c) The sequence $\{a_n + b_n\}$ satisfies

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Sometimes true: If $\{a_n\}$ and $\{b_n\}$ converge, the given statement is true using the limit laws.

Take $a_n = (-1)^n$, $b_n = (-1)^{n+1}$. Both $\{a_n\}$ and $\{b_n\}$ diverge so $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ doesn't exist. but $a_n + b_n = 0$ for all n. So $\lim_{n\to\infty} a_n + b_n = 0$

(d) The series $\sum a_n$ and $\sum b_n$ converge and $\sum a_n - b_n$ diverges.

Never true: According to the limit laws, if $\sum a_n$ and $\sum b_n$ converge, then $\sum a_n - b_n$ converges and

$$\sum a_n - b_n = \sum a_n - \sum b_n.$$

7. Find the value of c such that $\sum_{n=1}^{\infty} e^{nc} = 10$

The series $\sum_{n=1}^{\infty} e^{nc}$ is a geometric series with

first term e^c and common ratio e^c. So

$$\sum_{n=1}^{\infty} e^{nc} = \frac{e^c}{1-e^c} = 10$$

$$e^c = 10 - 10e^c$$

$$11e^c = 10$$

$$e^c = \frac{10}{11}$$

$$c = \ln\left|\frac{10}{11}\right|.$$

8. Determine if the following series converge or diverge. If a series converges, determine what it converges to.

(a) $\sum_{n=1}^{\infty} \frac{-2}{n^2+n}$

$$\frac{-2}{n^2+n} = \frac{A}{n+1} + \frac{B}{n} = \frac{2}{n+1} - \frac{2}{n}$$

$$-2 = An + Bn + B$$

$0 = A + B$ $\qquad$ $0 = A - 2$

$-2 = B$ $\qquad$ $A = 2$

$$S_n = \left(\cancel{\frac{2}{2}} - \frac{2}{1}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{2}}\right) + \left(\cancel{\frac{2}{4}} - \cancel{\frac{2}{3}}\right) + \cdots + \left(\frac{2}{n+1} - \cancel{\frac{2}{n}}\right)$$

$$= -2 + \frac{2}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{-2}{n^2+n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} -2 + \frac{2}{n+1} = -2$$

The series $\sum_{n=1}^{\infty} \frac{-2}{n^2+n}$ converges and the sum is -2.

(b) $\sum_{n=1}^{\infty} \ln(n)$

Since $\lim_{n\to\infty} \ln(n) = \infty$, the series $\sum_{n=1}^{\infty} \ln(n)$ diverges by the divergence test.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{\pi^{n+3}}$

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{\pi^{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\pi^3}\left(\frac{-2}{\pi}\right)^n$$

This series is geometric with first term $\frac{1}{\pi^3}$ and common ratio $\frac{-2}{\pi}$. Since $\left|\frac{-2}{\pi}\right| < 1$, the series will converge and

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{\pi^{n+3}} = \frac{\frac{1}{\pi^3}}{1+\frac{2}{\pi}}.$$

9. Determine if the following series are absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

$\displaystyle\lim_{n\to\infty} \frac{(-2)^n}{n^2}$ doesn't exist so by the test for divergence, $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$

Let $a_n = \frac{1}{1+n^2}$ and note $|a_n| = a_n > 0$.
Consider $b_n = \frac{1}{n^2} > 0$.

$$\lim_{n\to\infty} \frac{\frac{1}{1+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1 > 0$$

Thus, by the LCT, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Thus $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ is absolutely convergent.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$

$$L = \lim_{n\to\infty} \left| \frac{(-5)^{n+1}}{(n+1)!} \bigg/ \frac{(-5)^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5}{n+1} \right|$$

$$= 0 < 1$$

By the ratio test, since $L < 1$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$ is absolutely convergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

First, consider $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This series diverges by the p-test ($p = \frac{1}{2} \leq 1$). So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not absolutely convergent.

Note that $\dfrac{1}{\sqrt{n+1}} \leq \dfrac{1}{\sqrt{n}}$ for all n and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ so by the alternating series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent.

(e) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$

Note that $\ln\left(\frac{n+2}{n}\right) > 0$ for $n \geq 1$.

$\ln\left(\frac{n+2}{n}\right) = \ln(n+2) - \ln(n)$

$$S_n = \left(\cancel{\ln(3)} - \ln(1)\right) + \left(\cancel{\ln(4)} - \ln(2)\right) + \left(\cancel{\ln(5)} - \cancel{\ln(3)}\right) + \left(\ln(6) - \cancel{\ln(4)}\right)$$
$$+ \cdots + \left(\cancel{\ln(n)} - \cancel{\ln(n-2)}\right) + \left(\ln(n+1) - \cancel{\ln(n-1)}\right)$$
$$+ \left(\ln(n+2) - \cancel{\ln(n)}\right)$$
$$= -\ln(2) + \ln(n+1) + \ln(n+2)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(2) + \ln(n+1) + \ln(n+2) = \infty$$

So $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$ diverges.

(f) $\sum_{n=1}^{\infty} a_n$, $a_1 = 2$, $a_{n+1} = \left(\frac{5n+1}{4n+3}\right) a_n$

Consider $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{\left(\frac{5n+1}{4n+3}\right) a_n}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{5n+1}{4n+3}$$

$$= \frac{5}{4}$$

Since $L > 1$ and $a_n > 0$ for all n, the series $\sum_{n=1}^{\infty} a_n$ diverges.

(g) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Note that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1$). Further, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is alternating $\frac{1}{n+1} \leq \frac{1}{n}$, and $\lim_{n\to\infty} \frac{1}{n} = 0$ so by the alternating series test $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally.

(h) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3\cdot 4^n}{(-5)^{n+1}}+7^{-n}\right)$

$$\frac{3\cdot 4^n}{(-5)^{n+1}}+\frac{1}{7^n}=\frac{3}{-5}\left(\frac{-4}{5}\right)^n+\left(\frac{1}{7}\right)^n$$

Consider the series $\underbrace{\displaystyle\sum_{n=1}^{\infty}\frac{3}{-5}\left(\frac{-4}{5}\right)^n}_{①}$, $\underbrace{\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n}_{②}$

① $\displaystyle\sum_{n=1}^{\infty}\frac{-3}{5}\left(\frac{-4}{5}\right)^n$ is geometric with first term $\frac{12}{25}$ and common ratio $\frac{-4}{5}$.

Since $\left|\frac{-4}{5}\right|<1$, this series is absolutely convergent.

② $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n$ is geometric with first term $\frac{1}{7}$ and common ratio $\frac{1}{7}$. Since $\left|\frac{1}{7}\right|<1$, this series is absolutely convergent.

Since the above two series converge, we have that

$$\sum_{n=1}^{\infty}\left(\frac{3\cdot 4^n}{(-5)^{n+1}}+7^{-n}\right)=\sum_{n=1}^{\infty}\frac{-3}{5}\left(\frac{-4}{5}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n$$

which is the sum of two absolutely convergent series, and hence, is absolutely convergent.

10. How many terms should be added to estimate $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ to within 0.01 of its true value?

Note that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is alternating and

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}, \quad \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

Hence, by the alternating series remainder estimate we can bound the error E_n by

$$|E_n| \leq \frac{1}{\sqrt{n+1}}$$

So we can solve the inequality

$$0.01 \leq \frac{1}{\sqrt{n+1}}$$

$$\frac{1}{100} \leq \frac{1}{\sqrt{n+1}}$$

$$\frac{1}{10000} \leq \frac{1}{n+1}$$

$$10000 \geq n+1$$

$$9999 \geq n$$

We need at least 9999 terms to estimate $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ within 0.01.

11.

Time (s)	0	1	2	3	4	5
Velocity (ft/s)	1	1/2	1/5	1/10	1/17	1/27

(a) $V_n = \frac{1}{n^2+1}$

(b) $|R_n| < 0.1 \leq \int_n^\infty \frac{1}{x^2+1}dx$

* Note that $f(x) = \frac{1}{x^2+1}$ is positive, continuous, and decreasing on $[1,\infty)$. So we may use the integral test remainder estimate

$$\int_n^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_n^t \frac{1}{x^2+1}dx$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(x)$$

$$= \frac{\pi}{2} - \arctan(x)$$

Smallest n such that $\arctan(n) < \frac{\pi}{2} - 0.1$

(numerical check) $n=9$ does not satisfy the necessary inequality but $n=10$ does.

After 10 seconds, the pie will be within 0.1 ft of the total distance traveled.

(c) $S_{10} \approx 0.98178\ldots$

$$\int_{10}^\infty \frac{1}{x^2+1}dx \approx 0.09967$$

$$\int_{n+1}^\infty \frac{1}{x^2+1}dx \approx 0.09066$$

So if s is the true distance the pie will travel, we know

$$\int_{n+1}^{\infty} \frac{1}{x^2+1}\,dx + S_{10} \leq s \leq S_{10} + \int_{n}^{\infty} \frac{1}{x^2+1}\,dx$$

$$1.07244 \leq s \leq 1.08145$$

Taking the average of the bounds, we estimate that the pie travelled

$$s \approx 1.076945 \text{ ft.}$$